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Unitary groups as a complete invariant

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Abstract

Dye proved that the discrete unitary group in a factor determines the algebraic type of the factor. We show that if the unitary groups of two simple unital AH-algebras of slow dimension growth and of real rank zero are isomorphic as abstract groups, then their K_0 -ordered groups are isomorphic. Also, using Gong and Dadarlat's classification theorem, we prove that such C^* -algebras are isomorphic if and only if their unitary groups are isomorphic as topological groups. For simple, unital purely infinite C^* -algebras, we show that two unital Kirchberg algebras are $*$ -isomorphic if and only if their unitary groups are isomorphic as abstract groups.

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1. Introduction

In 1955, H. Dye proved that two von Neumann factors not of type I_{2n} are isomorphic (via a linear or a conjugate linear $*$ -isomorphism) if and only if their unitary groups are isomorphic as abstract groups.

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In this paper, we generalize Dye's approach for a large class of simple amenable, unital, separable C^* -algebras; from an isomorphism of the unitary groups of such algebras, we deduce an isomorphism of their K -theory.

Let A and B be two unital C^* -algebras, and φ be an isomorphism between their unitary groups. As φ preserves self-adjoint unitaries, it induces a natural bijection $\theta_\varphi : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ between the sets of projections of A and B given by

$$1 - 2\theta_\varphi(p) = \varphi(1 - 2p), \quad p \in \mathcal{P}(A).$$

This bijection θ_φ preserves unitary equivalence of projections, but is not necessarily a projection orthoisomorphism (i.e. a bijection map which preserves orthogonality of projections).

For simple C^* -algebras, we prove in Section 2 technical properties of the bijection θ_φ . In Section 3, if A is oddly decomposable (see Definition 3.1), we associate to θ_φ a partition $\{\mathcal{P}_e, \mathcal{P}_o\}$ of $\mathcal{P}(A) \setminus \{0, 1\}$ such that the map $\tilde{\theta}_\varphi : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ defined by

$$\tilde{\theta}_\varphi(q) = \begin{cases} \theta_\varphi(q) & \text{if } q \in \mathcal{P}_e, \\ 1 - \theta_\varphi(q) & \text{if } q \in \mathcal{P}_o, \\ 1 & \text{if } q = 1, \\ 0 & \text{if } q = 0 \end{cases}$$

is an orthoisomorphism and preserves the unitary equivalence of projection (Theorem 2.21 and Theorem 3.7).

In Section 4.1, we consider the class \mathcal{F}_1 of simple, unital, separable C^* -algebras of real rank zero which have cancellation and whose K_0 -groups are noncyclic and weakly unperforated. This class contains in particular all simple, unital AH-algebras of slow dimension growth, having real rank zero.

Let φ be as above and A and B be in class \mathcal{F}_1 . Then using Theorem 3.7, we show in Theorem 4.3 that there exists an order isomorphism from $K_0(A)$ to $K_0(B)$ sending $[1_A]$ to $[1_B]$. In particular, if A and B are both simple, unital AF-algebras, or both irrational rotation algebras and their unitary groups are isomorphic (as abstract groups), then A and B are isomorphic as C^* -algebras.

In Section 4.2, we deduce from Gong and Dadarlat's classification of simple, unital, AH-algebras of slow dimension growth and of real rank zero, that any two such algebras are isomorphic if and only if their unitary groups are isomorphic (as topological groups).

In Section 5, we study the case of simple, unital purely infinite C^* -algebras. We show that if A and B is a pair of such algebras, whose unitary groups are isomorphic (as abstract groups), then there is an isomorphism from $K_0(A)$ to $K_0(B)$, sending $[1_A]$ to $[1_B]$, and the groups $K_1(A)$ and $K_1(B)$ are isomorphic. Recall that a Kirchberg algebra is a purely infinite, simple, nuclear, separable C^* -algebra. From Kirchberg–Phillips's classification, we then deduce that two unital Kirchberg algebras belonging to the UCT-class \mathcal{N} are isomorphic if and only if their unitary groups are isomorphic (as abstract groups).

The research presented in this paper grew up from A. Booth's MSc thesis [5] and part of A. Al-Rawashdeh's PhD thesis [1]; the case of the simple, unital AF-algebras was obtained in [5], and those of simple, unital AT-algebras of real rank zero and of unital Kirchberg algebras were obtained in [1]. These results are generalized in this paper.

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2. Properties of the induced map θ_φ

Let A and B be two unital C^* -algebras. If $\varphi : \mathcal{U}(A) \rightarrow \mathcal{U}(B)$ is an isomorphism between the unitary groups of A and B , then φ maps the self-adjoint unitaries of A onto the self-adjoint unitaries of B , and defines a natural map $\theta = \theta_\varphi : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ by setting

$$1 - 2\theta(p) = \varphi(1 - 2p), \quad p \in \mathcal{P}(A).$$

The following properties of the map θ can be easily checked.

Proposition 2.1. (See [9].) *Let A and B be unital C^* -algebras, $\varphi : \mathcal{U}(A) \rightarrow \mathcal{U}(B)$ be a group isomorphism and θ be the induced map between the projections. Then*

- (i) $\theta(upu^*) = \varphi(u)\theta(p)\varphi(u)^*$,
- (ii) $\theta(0) = 0$,
- (iii) if $p, q \in \mathcal{P}(A)$ commute, then so do $\theta(p)$ and $\theta(q)$ in $\mathcal{P}(B)$,
- (iv) $\theta(p\Delta q) = \theta(p)\Delta\theta(q)$, where Δ denotes the symmetric difference of commuting projections i.e. $p\Delta q = p + q - 2pq$.

If the center $\mathcal{Z}(B)$ of a unital C^* -algebra B is reduced to the scalars, and $\varphi : \mathcal{U}(A) \rightarrow \mathcal{U}(B)$ is as above, then $\varphi(-1) = -1$. Indeed, note that -1 is a central, self-adjoint unitary which is not 1, so the same is true for $\varphi(-1)$. As a consequence, we get.

Lemma 2.2. *Let A and B be unital C^* -algebras, whose center $\mathcal{Z}(B) = \mathbb{C}1$. Let $\varphi : \mathcal{U}(A) \rightarrow \mathcal{U}(B)$ be a group isomorphism and $\theta : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ be as above. Then $\theta(1) = 1$, and for each $p \in \mathcal{P}(A)$, $\theta(1 - p) = 1 - \theta(p)$.*

In the following lemma, we collect equalities that we will use often.

Lemma 2.3. *Let A , B , φ and θ be as in Lemma 2.2. If p and q are commuting projections of A and if $r = p\Delta q \in \mathcal{P}(A)$ denotes their symmetric difference, then*

$$\theta(p)\theta(1 - p) = \theta(p)\theta(r), \tag{1}$$

$$\theta(1 - p)\theta(q) = \theta(q)\theta(r), \tag{2}$$

$$\theta(p)\theta(q) = \theta(p)\theta(1 - r) = \theta(q)\theta(1 - r) = \theta(p)\theta(q)\theta(1 - r), \tag{3}$$

$$\theta(1 - p)\theta(1 - q)\theta(r) = 0, \tag{4}$$

$$\theta(r) = \theta(p)\theta(1 - q) + \theta(1 - p)\theta(q), \tag{5}$$

$$\theta(1 - r) = \theta(p)\theta(q) + \theta(1 - p)\theta(1 - q). \tag{6}$$

Proof. By Lemma 2.2, we have $\theta(p)\theta(r) = \theta(p)(1\Delta\theta(q)) = \theta(p)\theta(1 - q)$ and this shows (1) and (2). Using this we get $\theta(p)\theta(1 - r) = \theta(p) - \theta(p)\theta(1 - q) = \theta(p)\theta(q)$. Also, as

$$\theta(p)\theta(q) = \theta(p)(\theta(p)\theta(q)) = \theta(p)\theta(q)\theta(1 - r),$$

we have (3). As

$$\theta(1-p)\theta(1-q)\theta(r) = [\theta(1-p)\theta(1-q)\theta(p)]\Delta[\theta(1-p)\theta(1-q)\theta(q)] = 0,$$

(4) follows. By (1) and (2), we have $\theta(p)\theta(1-q) \leq \theta(r)$ and $\theta(1-p)\theta(q) \leq \theta(r)$ and by (3) and (4), we have $\theta(p)\theta(q) \leq \theta(1-r)$ and $\theta(1-p)\theta(1-q) \leq \theta(1-r)$. Since

$$\theta(p)\theta(1-q) + \theta(1-p)\theta(q) + \theta(p)\theta(q) + \theta(1-p)\theta(1-q) = 1$$

the last two assertions follow. \square

To simplify notations, let us introduce the following:

Notation 2.4. (i) The quadruple (A, B, φ, θ) will denote a pair of simple unital C^* -algebras A and B , a group isomorphism $\varphi: \mathcal{U}(A) \rightarrow \mathcal{U}(B)$ and $\theta: \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ the induced bijection.

(ii) Let $\widetilde{\mathcal{P}}(A)$ denote the set $\mathcal{P}(A) \setminus \{0, 1\}$.

The following lemma is a generalization to simple, unital C^* -algebras of Lemma 10 of [9].

Lemma 2.5. Let (A, B, φ, θ) be as in (2.4). Then for each fixed $p \in \widetilde{\mathcal{P}}(A)$, there exist characters a_p and b_p of the circle group \mathbb{S}^1 such that

$$\varphi(\mu p + 1 - p) = a_p(\mu)\theta(p) + b_p(\mu)(1 - \theta(p)), \quad \text{for all } \mu \in \mathbb{S}^1.$$

Proof. Let us fix $p \in \widetilde{\mathcal{P}}(A)$. If $\theta(p)'$ denotes the commutant of $\theta(p)$ in B , we then have:

$$\theta(p)' = \theta(p)B\theta(p) + (1 - \theta(p))B(1 - \theta(p)).$$

As B is simple and $\theta(p)B\theta(p)$ is a hereditary C^* -subalgebra of B , then $\theta(p)B\theta(p)$ is also simple and therefore its center $\mathcal{Z}(\theta(p)B\theta(p)) = \mathbb{C}(1 - \theta(p))$. Similarly $\mathcal{Z}((1 - \theta(p))B(1 - \theta(p))) = \mathbb{C}\theta(p)$. Therefore, we have

$$\mathcal{Z}(\theta(p)') = \{\mu\theta(p) + \nu(1 - \theta(p)); \mu, \nu \in \mathbb{C}\}.$$

As any element of B is a linear combinations of four unitaries, any element in the center of the unitaries of $B \cap \theta(p)'$ belongs to the center of $B \cap \theta(p)'$. Therefore we have

$$\mathcal{Z}(\mathcal{U}(B) \cap \theta(p)') = \{\mu\theta(p) + \nu(1 - \theta(p)); \mu, \nu \in \mathbb{S}^1\}.$$

Now, as for any $\mu \in \mathbb{S}^1$, the unitary $\mu p + 1 - p$ belongs to the center of $\mathcal{U}(p' \cap A)$, then $\varphi(\mu p + 1 - p) \in \mathcal{Z}(\mathcal{U}(B) \cap \theta(p)')$. Hence, as φ is an isomorphism, there exist characters a_α and b_α of \mathbb{S}^1 such that

$$\varphi(\mu p + 1 - p) = a_p(\mu)\theta(p) + b_p(\mu)(1 - \theta(p)), \quad \text{for all } \mu \in \mathbb{S}^1. \quad \square$$

Lemma 2.6. Let (A, B, φ, θ) be as in (2.4). Let $p, q \in \widetilde{\mathcal{P}(A)}$ be such that $pq = 0$ and $p + q = r \neq 1$. Then

$$b_q a_p = a_r \quad \text{if } \theta(p)\theta(1-q) \neq 0, \quad (7)$$

$$b_p a_q = a_r \quad \text{if } \theta(1-p)\theta(q) \neq 0, \quad (8)$$

$$a_p a_q = b_r \quad \text{if } \theta(p)\theta(q) \neq 0, \quad (9)$$

$$b_p b_q = b_r \quad \text{if } \theta(1-p)\theta(1-q) \neq 0. \quad (10)$$

Proof. By Lemma 2.5, we have

$$\begin{aligned} \varphi(\mu p + 1 - p)\varphi(\mu q + 1 - q) &= a_p a_q \theta(p)\theta(q) + a_p b_q \theta(p)\theta(1 - q) \\ &\quad + b_p a_q \theta(1 - p)\theta(q) + b_p b_q \theta(1 - p)\theta(1 - q), \end{aligned}$$

for all $\mu \in \mathbb{S}^1$. By Lemma 2.3(5), (6) and Lemma 2.5 we have

$$\begin{aligned} \varphi(\mu r + 1 - r) &= a_r(\mu)\theta(r) + b_r(\mu)\theta(1 - r) \\ &= a_r \theta(p)\theta(1 - q) + a_r \theta(1 - p)\theta(q) \\ &\quad + b_r \theta(p)\theta(q) + b_r \theta(1 - p)\theta(1 - q). \end{aligned}$$

Therefore using Lemma 2.2 and by identifying coefficients, we get Eqs. (7) to (10). \square

To each $p \in \widetilde{\mathcal{P}(A)}$, we associate the pair of characters (a_p, b_p) and the character $c_p = a_p \bar{b}_p \in \widehat{\mathbb{S}^1}$.

Let us denote by \mathfrak{C} the equivalence relation on $\widetilde{\mathcal{P}(A)}$, given by:

$$p \mathfrak{C} q \quad \text{iff} \quad c_p = c_q.$$

Introducing the following notation

$$(a_p, b_p) = (a_q, b_q) \quad \text{iff} \quad a_p b_q = b_p a_q \quad \text{iff} \quad c_p = c_q,$$

we have:

$$p \mathfrak{C} q \quad \text{iff} \quad (a_p, b_p) = (a_q, b_q) \quad \text{iff} \quad c_p = c_q.$$

Proposition 2.7. Let (A, B, φ, θ) be as in (2.4). For any $p \in \widetilde{\mathcal{P}(A)}$:

- (i) If $q \in \widetilde{\mathcal{P}(A)}$ is unitarily equivalent to p , then $a_p = a_q$ and $b_p = b_q$, hence $p \mathfrak{C} q$.
- (ii) $a_p^2(\mu) \neq b_p^2(\mu)$, for all $\mu \in \mathbb{S}^1$, $\mu \neq 1$.
- (iii) $c_p = c_{1-p}$, hence $p \mathfrak{C} (1 - p)$.

Proof. (i) Let $u \in \mathcal{U}(A)$ be such that $p = uqu^*$. Then

$$\begin{aligned} a_p(\mu)\theta(p) + b_p(\mu)\theta(1-p) &= \varphi(\mu p + 1 - p) \\ &= \varphi(\mu uqu^* + 1 - uqu^*) \\ &= \varphi(u)(a_q(\mu)\theta(q) + b_q(\mu)\theta(1-q))\varphi(u)^* \\ &= a_q(\mu)\theta(p) + b_q(\mu)\theta(1-p). \end{aligned}$$

So $a_p = a_q$ and $b_p = b_q$.

(ii) If $a_p^2(\mu) = b_p^2(\mu)$, for some $\mu \neq 1$, then by Lemma 2.5 we have

$$\begin{aligned} \varphi(\mu^2 p + 1 - p) &= [a_p(\mu)\theta(p) + b_p(\mu)\theta(1-p)][a_p(\mu)\theta(p) + b_p(\mu)\theta(1-p)] \\ &= a_p^2(\mu)\theta(p) + b_p^2(\mu)\theta(1-p) \\ &= a_p^2(\mu)1. \end{aligned}$$

Therefore, $\varphi(\mu^2 p + 1 - p)$ is a central element in B , applying φ^{-1} we get that $\mu^2 p + 1 - p$ is also central in A , hence equals to a scalar multiple of 1, which gives a contradiction.

(iii) As $\mu 1 = (\mu p + 1 - p)(\mu(1 - p) + p)$, we have

$$\varphi(\mu 1) = a_p(\mu)b_{1-p}(\mu)\theta(p) + b_p(\mu)a_{1-p}(\mu)(1 - \theta(p)),$$

hence $a_p(\mu)b_{1-p}(\mu) = b_p(\mu)a_{1-p}(\mu)$, for all $\mu \in \mathbb{S}^1$. \square

For any two commuting projections p and q of $\widetilde{\mathcal{P}(A)}$, let $\mathcal{S}_{p,q}$ denote the set $\{\theta(p)\theta(q), \theta(p)\theta(1-q), \theta(1-p)\theta(q), \theta(1-p)\theta(1-q)\}$. By Proposition 2.1(iii) and Lemma 2.2, the set $\mathcal{S}_{p,q}$ consists of mutually orthogonal projections which form a partition of 1.

Proposition 2.8. Let (A, B, φ, θ) be as in (2.4). Let $p, q \in \widetilde{\mathcal{P}(A)}$ be such that $pq = 0$ and $p+q = r \neq 1$. Then exactly one element of $\mathcal{S}_{p,q}$ is zero.

Proof. If all elements of $\mathcal{S}_{p,q}$ were non-zero, then by Lemma 2.6 we would have $a_r^2 = b_q a_p b_p a_q$ and $b_r^2 = a_p a_q b_p b_q$, hence $a_r^2(\mu) = b_r^2(\mu)$, for all $\mu \in \mathbb{S}^1$; but this is impossible by Proposition 2.7(ii). Suppose that $\theta(p)\theta(q) = 0$. If $\theta(p)\theta(1-q) = 0$, then this implies that $\theta(p) = 0$ which contradicts the injectivity of θ , similarly $\theta(1-p)\theta(q) \neq 0$. If $\theta(1-p)\theta(1-q) = 0$, then

$$\begin{aligned} 1 &= \theta(p) + \theta(q) \\ &= \theta(p\Delta q) \\ &= \theta(r), \end{aligned}$$

which again contradicts the injectivity of the map θ . Suppose that $\theta(p)\theta(1-q) = 0$, hence $\theta(p)\theta(q) \neq 0$. If $\theta(1-p)\theta(q) = 0$, then $\theta(p) = \theta(q)$, which contradicts the injectivity of θ . If $\theta(1-p)\theta(1-q) = 0$, then $1 - \theta(q) - (\theta(p) - \theta(p)\theta(q)) = 0$, so this implies that $\theta(q) = 1$, which leads to a contradiction. Finally if we assume that $\theta(1-p)\theta(1-q) = 0$, then by similar techniques we show that none of the other three elements of $\mathcal{S}_{p,q}$ is zero, hence the proposition is checked. \square

Combining Lemma 2.6 and Proposition 2.8, we get the following fundamental theorem.

Theorem 2.9. *Let (A, B, φ, θ) be as in (2.4). Let $p, q \in \widetilde{\mathcal{P}(A)}$ be such that $pq = 0$ and $p + q = r \neq 1$. Then*

$$\theta(p)\theta(q) = 0 \quad \Leftrightarrow \quad c_p = c_q = c_r, \quad (11)$$

$$\theta(1-p)\theta(1-q) = 0 \quad \Leftrightarrow \quad c_p = c_q = \bar{c}_r, \quad (12)$$

$$\theta(1-p)\theta(q) = 0 \quad \Leftrightarrow \quad c_p = \bar{c}_q = c_r, \quad (13)$$

$$\theta(p)\theta(1-q) = 0 \quad \Leftrightarrow \quad \bar{c}_p = c_q = c_r. \quad (14)$$

Proof. Suppose $\theta(p)\theta(q) = 0$. Then $b_p a_q = a_r = b_q a_p$ and $b_r = b_p b_q$, therefore $a_r b_p = b_r a_p$ which implies $p \mathfrak{C} q$ and $q \mathfrak{C} r$. Conversely, if $\theta(p)\theta(1-q) = 0$, then $b_p a_q = a_r$, $a_p a_q = b_r$ and $b_p b_q = b_r$, implies $a_p^2 = b_p^2$ which contradicts Proposition 2.7(ii). If $\theta(1-p)\theta(q) = 0$, then $b_q a_p = a_r$, $a_p a_q = b_r$ and $b_p b_q = b_r$, which is impossible as $a_q^2 = b_q^2$. If $\theta(1-p)\theta(1-q) = 0$, then $b_q a_p = a_r$, $b_p a_q = a_r$ and $a_p a_q = b_r$, implies $b_p^2 = a_p^2$ which is also impossible again by Proposition 2.7(ii), hence Eq. (11) follows. To prove the third equation, suppose that $\theta(1-p)\theta(q) = 0$. Then Eqs. (7), (9) and (10) prove that $c_p = \bar{c}_q = c_r$. Conversely, if $\theta(p)\theta(q) = 0$, then $a_q^2 = b_q^2$ which gives a contradiction by Proposition 2.7(ii). Similar contradictions can be deduced by assuming $\theta(p)\theta(1-q) = 0$ or $\theta(1-p)\theta(1-q) = 0$, therefore $\theta(1-p)\theta(q) = 0$. The other two equations are checked in the same way. \square

Corollary 2.10. *Let (A, B, φ, θ) be as in (2.4), and $p, q \in \widetilde{\mathcal{P}(A)}$ with $pq = 0$ and $p + q \neq 1$. Then either $c_p = c_q$ or $c_p = \bar{c}_q$. Moreover, if $p \mathfrak{C} q$ and $\theta(p)\theta(q) \neq 0$, then*

$$\theta(p)\theta(q) = \theta(1-p)\Delta\theta(q).$$

Proof. The first statement comes by inspecting the result of Theorem 2.9. Now notice that

$$\begin{aligned} \theta(1-p)\theta(1-q) &= (1-\theta(p))(1-\theta(q)) \\ &= 1 - \theta(p) - \theta(q) + \theta(p)\theta(q) \end{aligned}$$

and

$$(1-\theta(p))\Delta\theta(q) = 1 - \theta(p)\Delta\theta(q).$$

Then we have

$$\begin{aligned} \theta(1-p)\Delta\theta(q) &= 1 - (\theta(p)\Delta\theta(q)) \\ &= 1 - \theta(p) - \theta(q) + 2\theta(p)\theta(q) \\ &= \theta(1-p)\theta(1-q) + \theta(p)\theta(q), \end{aligned}$$

from which the second statement follows immediately. \square

Corollary 2.11. Let (A, B, φ, θ) be as in (2.4), and $p, q \in \widetilde{\mathcal{P}}(A)$ be commuting projections. Then either $(a_p, b_p) = (a_q, b_q)$ or $(a_p, b_p) = (b_q, a_q)$.

Proof. If $pq = 0$, then the result comes directly from Theorem 2.9. If $pq = p$, then $p(1-q) = 0$, so by Theorem 2.9, either $(a_p, b_p) = (a_{1-q}, b_{1-q}) = (a_q, b_q)$ or $(a_p, b_p) = (b_{1-q}, a_{1-q}) = (b_q, a_q)$. Similarly the case $pq = q$. If $0, p, q$ and pq are distinct, then

$$0 = pq(1-p) = (1-q)pq.$$

Then by Theorem 2.9, we have

$$(a_{pq}, b_{pq}) = (a_{1-p}, b_{1-p}) = (a_p, b_p) \quad \text{or} \quad (a_{pq}, b_{pq}) = (b_{1-p}, a_{1-p}) = (b_p, a_p).$$

Also,

$$(a_{pq}, b_{pq}) = (a_{1-q}, b_{1-q}) = (a_q, b_q) \quad \text{or} \quad (a_{pq}, b_{pq}) = (b_{1-q}, a_{1-q}) = (b_q, a_q),$$

hence the corollary is checked. \square

Remark 2.12. Let (A, B, φ, θ) be as in (2.4). If any two projections of $\widetilde{\mathcal{P}}(A)$ are \mathfrak{C} -equivalent, then by Theorem 2.9, θ is an orthoisomorphism.

Proposition 2.13. Let (A, B, φ, θ) be as in (2.4), and suppose that for each pair of projections $p, q \in \widetilde{\mathcal{P}}(A)$ there exist unitarily equivalent non-zero subprojections $p_1 \leq p$ and $q_1 \leq q$. Then the quotient space $\widetilde{\mathcal{P}}(A)/\mathfrak{C}$ has at most two elements. More precisely, for any projection $p \in \widetilde{\mathcal{P}}(A)$, we have $c(\widetilde{\mathcal{P}}(A)) \subseteq \{c_p, \bar{c}_p\}$.

Proof. Let $p, q \in \widetilde{\mathcal{P}}(A)$. By assumption there exist $p_1 \leq p$ and $q_1 \leq q$ such that $p_1 \mathfrak{C} q_1$ and then by Proposition 2.7(i), $(a_{p_1}, b_{p_1}) = (a_{q_1}, b_{q_1})$. As p commutes with p_1 , $c_{p_1} = c_p$ or $c_{p_1} = \bar{c}_p$. Similarly for q and q_1 . So fixing p , for any projection $q \in \widetilde{\mathcal{P}}(A)$ we have $c_q = c_p$ or $c_q = \bar{c}_p$. \square

Lemma 2.14. Let (A, B, φ, θ) be as in (2.4). Let p_1, p_2, p_3 be three \mathfrak{C} -equivalent, pairwise orthogonal, non-trivial projections of A with $\sum p_i = r \neq 1$.

- (a) If $\theta(p_1)$ is not orthogonal to $\theta(p_2)$ and $\theta(p_3)$, then $\theta(p_2)\theta(p_3) \neq 0$.
- (b) If $\theta(p_1)$ is orthogonal to $\theta(p_2)$ and $\theta(p_3)$, then $\theta(p_2)\theta(p_3) = 0$.

Proof. (a) Let $p = p_1$ and $q = p_2 + p_3$. If $\theta(p_2)\theta(p_3) = 0$, then by Theorem 2.9, $c_{p_2} = c_{p_3} = c_p$, hence $p \mathfrak{C} q$. By Proposition 2.8 and Theorem 2.9, applied to p and q , either

$$\theta(p)\theta(q) = 0 \quad \text{or} \quad \theta(1-p)\theta(1-q) = 0.$$

If $\theta(p)\theta(q) = 0$, then by the second part of Corollary 2.10, we have

$$\begin{aligned}
0 &= \theta(p_1)\theta(p_2 + p_3) = \theta(p_1)[\theta(p_2)\Delta\theta(p_3)] \\
&= [\theta(p_1)\theta(p_2)]\Delta[\theta(p_1)\theta(p_3)] \\
&= [\theta(1 - p_1)\Delta\theta(p_2)]\Delta[\theta(1 - p_1)\Delta\theta(p_3)] \\
&= \theta(p_2)\Delta\theta(p_3) \\
&= \theta(p_2 + p_3).
\end{aligned}$$

This contradicts the fact that $p_2 + p_3 \neq 0$ and θ is a bijection.

If $\theta(1 - p)\theta(1 - q) = 0$, then as above we have

$$\begin{aligned}
0 &= \theta(1 - p)\theta(1 - q) \\
&= \theta(1 - p_1)\theta(1 - (p_2 + p_3)) = (1 - \theta(p_1))(1 - \theta(p_2)\Delta\theta(p_3)) \\
&= 1 - \theta(p_1) - \theta(p_1)\Delta\theta(p_3) + \theta(p_1)\theta(p_1)\Delta\theta(p_1)\theta(p_3) \\
&= 1 - \theta(p_1).
\end{aligned}$$

But as $p_1 \neq 1$, then $1 - \theta(p_1) \neq 0$ and therefore $\theta(p_2)\theta(p_3) \neq 0$.

(b) By assumption $\theta(p_1) \leq \theta(1 - p_2)$ and $\theta(p_1) \leq \theta(1 - p_3)$ and $\theta(p_1) \leq \theta(1 - p_2)\theta(1 - p_3)$. Assume $\theta(p_2)\theta(p_3) \neq 0$. As $p_2 \not\mathfrak{C} p_3$, by Theorem 2.9, $\theta(1 - p_2)\theta(1 - p_3) = 0$ and therefore $\theta(p_1) = 0$, which would contradict the injectivity of θ . \square

Lemma 2.15. Let (A, B, φ, θ) be as in (2.4). Let $\{p_i\}_{i=1}^n$ be n pairwise orthogonal, \mathfrak{C} -equivalent, non-trivial projections of A such that $p_i + p_j + p_k \neq 1$ for any $1 \leq i < j < k \leq n$. If $\theta(p_k)\theta(p_l) = 0$ for some $1 \leq k, l \leq n$, then $\theta(p_i)\theta(p_j) = 0$ for all $1 \leq i \neq j \leq n$.

Proof. The proof is done by induction. The cases $n = 1$ and $n = 2$ are trivial and the case $n = 3$ follows from Lemma 2.14 since $p_1 + p_2 + p_3 \neq 1$. Suppose now that the result is true for some $m \geq 3$. Let $\{p_i\}_{i=1}^{m+1}$ be $m + 1$ projections of A satisfying the assumptions of the lemma. Without loss of generality, we can suppose that $\theta(p_1)\theta(p_2) = 0$. By induction hypothesis, $\theta(p_i)\theta(p_j) = 0$ for all $1 \leq i \neq j \leq m$; so it remains to show that $\theta(p_i)\theta(p_{m+1}) = 0$ for all $1 \leq i \leq m$. Since $m > 3$, both the collections $\{p_2, p_3, \dots, p_{m+1}\}$ and $\{p_1, p_3, \dots, p_{m+1}\}$ satisfy the induction hypothesis, so $\theta(p_i)\theta(p_{m+1}) = 0$, for all $1 \leq i \leq m$. \square

Lemma 2.16. Let (A, B, φ, θ) be as in (2.4). Let p_1, p_2, p_3 be three pairwise orthogonal, \mathfrak{C} -equivalent, non-trivial projections of A such that $r = \sum_{i=1}^3 p_i \neq 1$. Then $p_1 \mathfrak{C} r$.

Proof. By Lemma 2.15, either $\theta(p_i)\theta(p_j) = 0$ for all $1 \leq i \neq j \leq 3$ or $\theta(p_i)\theta(p_j) \neq 0$, for all $1 \leq i, j \leq 3$. In the first case we get that

$$\begin{aligned}
\theta(p_1 + p_2)\theta(p_3) &= [\theta(p_1)\Delta\theta(p_2)]\theta(p_3) \\
&= (\theta(p_1) + \theta(p_2))\theta(p_3) \\
&= \theta(p_1)\theta(p_3) + \theta(p_2)\theta(p_3) \\
&= 0.
\end{aligned}$$

Using Theorem 2.9, we have $(p_1 + p_2) \mathfrak{C} p_3 \mathfrak{C} (p_1 + p_2 + p_3)$. Hence $r \mathfrak{C} p_1$.

In the second case, by Theorem 2.9, $c_{p_i} = c_{p_j} = \overline{c_{p_i+p_j}}$ and by Corollary 2.10,

$$\theta(p_i)\theta(p_j) = \theta(1 - p_i)\Delta\theta(p_j)$$

for all $1 \leq i \neq j \leq 3$. Let us assume that r is not \mathfrak{C} -equivalent to p_1 and derive a contradiction. For $p = p_1$ and $q = p_2 + p_3$, we have

$$c_p = c_{p_3} = c_{p_2} = \overline{c_{p_2+p_3}} = \overline{c_q}.$$

By Theorem 2.9, we then obtain $\theta(p)\theta(1 - q) = 0$; hence

$$\begin{aligned} \theta(p_1) &= \theta(p_1)\theta(p_2 + p_3) = \theta(p_1)\theta(p_2\Delta p_3) \\ &= [\theta(p_1)\theta(p_2)]\Delta[\theta(p_1)\theta(p_3)] \\ &= [\theta(1 - p_1)\Delta\theta(p_2)]\Delta[\theta(1 - p_1)\Delta\theta(p_3)] \\ &= \theta(p_2)\Delta\theta(p_3) = \theta(p_2 + p_3), \end{aligned}$$

which contradicts the injectivity of θ . So, $c_r = c_{p_1}$. \square

Corollary 2.17. *Let (A, B, φ, θ) be as in (2.4). If p_1, p_2 and p_3 are three pairwise orthogonal, non-trivial projections of A , such that $p = \sum_{i=1}^3 p_i \neq 1$, then p is \mathfrak{C} -equivalent to some p_i (equivalently $c_p = c_{p_i}$ for some i).*

Proof. By Corollary 2.10, for $1 \leq i \leq 3$, we have $c_{p_i} = c_{p_1}$ or \bar{c}_{p_1} and by Lemma 2.16, we can assume that $c_{p_i} = \bar{c}_{p_1}$ for $i = 2$ or 3 , and without loss of generality that $c_{p_1} = c_{p_2} = \bar{c}_{p_3}$. By Theorem 2.9, we get that $c_{p_1+p_2} = c_{p_1}$ or \bar{c}_{p_1} , and by the same theorem applied to $p_1 + p_2$ and p_3 , we get

$$c_p = c_{p_1+p_2+p_3} = c_{p_1} \quad \text{or} \quad \bar{c}_{p_1}.$$

Hence $c_p = c_{p_1}$ or c_{p_3} . \square

Extending by induction Lemma 2.16 and Corollary 2.17 we get:

Proposition 2.18. *Let (A, B, φ, θ) be as in (2.4), and n be an odd integer. Let $\{p_i\}_{i=1}^n$ be n pairwise orthogonal, \mathfrak{C} -equivalent, non-trivial projections of A with $\sum_{i=1}^n p_i = r \neq 1$. Then $p_i \mathfrak{C} r$, for $1 \leq i \leq n$.*

Corollary 2.19. *Let (A, B, φ, θ) be as in (2.4), and n be an odd integer. Let $\{p_i\}_{i=1}^n$ be n pairwise orthogonal, non-trivial projections of A with $p = \sum_{i=1}^n p_i \neq 1$. Then p is \mathfrak{C} -equivalent to p_i , for some i (equivalently $c_p = c_{p_i}$ for some i).*

Before stating Theorem 2.21, let us introduce the following definition:

Definition 2.20. Let $\theta : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ be any map. If S is a subset of $\mathcal{P}(A)$, then θ is said to preserve orthogonality on S (respectively flip orthogonality on S) if $\theta(p)\theta(q) = 0$ (respectively $(1 - \theta(p))(1 - \theta(q)) = 0$), for any orthogonal projections p and q in S .

Theorem 2.21. Let (A, B, φ, θ) be as in (2.4) and suppose that the cardinality of $\widetilde{\mathcal{P}(A)}/\mathfrak{C}$ is two. Let $\{\mathcal{P}_e, \mathcal{P}_o\}$ be the partition of $\widetilde{\mathcal{P}(A)}$ with respect to the equivalence relation \mathfrak{C} . If θ preserves orthogonality on \mathcal{P}_e and flips orthogonality on \mathcal{P}_o , then the map $\tilde{\theta} : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ defined by

$$\tilde{\theta}(q) = \begin{cases} \theta(q) & \text{if } q \in \mathcal{P}_e, \\ 1 - \theta(q) & \text{if } q \in \mathcal{P}_o, \\ 1 & \text{if } q = 1, \\ 0 & \text{if } q = 0 \end{cases}$$

is an orthoisomorphism and $\tilde{\theta}(uqu^*) = \varphi(u)\tilde{\theta}(q)\varphi(u)^*$, for $q \in \mathcal{P}(A)$, $u \in \mathcal{U}(A)$.

To prove of Theorem 2.21, we need the following lemma:

Lemma 2.22. Let (A, B, φ, θ) and $\{\mathcal{P}_e, \mathcal{P}_o\}$ be as in Theorem 2.21 (i.e. θ preserves orthogonality on \mathcal{P}_e and flips orthogonality on \mathcal{P}_o). If p and q are orthogonal projections in $\widetilde{\mathcal{P}(A)}$ such that $p + q = r \neq 1$, then

1. $p, q \in \mathcal{P}_e \Rightarrow r \in \mathcal{P}_e$,
2. $p, q \in \mathcal{P}_o \Rightarrow r \in \mathcal{P}_e$,
3. $p \in \mathcal{P}_e, q \in \mathcal{P}_o \Rightarrow r \in \mathcal{P}_o$.

Proof. Parts (1) and (2) follow directly from Theorem 2.9. For (3), let $p \in \mathcal{P}_e, q \in \mathcal{P}_o$ and suppose that $r \in \mathcal{P}_e$. Then $1 - r \in \mathcal{P}_e$. So $(1 - r) + p \in \mathcal{P}_e$ from (1). But then $q = 1 - (1 - r + p) \in \mathcal{P}_e$, which gives a contradiction. This proves 2.22. \square

Proof of Theorem 2.21. As $\{\mathcal{P}_e, \mathcal{P}_o\}$ forms a partition of $\widetilde{\mathcal{P}(A)}$ and by Lemma 2.2, $\tilde{\theta}$ is a bijection. We prove that $\tilde{\theta}$ preserves orthogonality in a case by case study. Suppose that p and q are non-trivial orthogonal projections of A . By Lemma 2.2, we can assume that $p + q \neq 1$. If $p, q \in \mathcal{P}_e$ (respectively $p, q \in \mathcal{P}_o$), then by assumption $\theta(p)\theta(q) = 0$ (respectively $\theta(1 - p)\theta(1 - q) = 0$), and therefore $\tilde{\theta}(p)\tilde{\theta}(q) = 0$.

If $p \in \mathcal{P}_e$ and $q \in \mathcal{P}_o$, then by Lemma 2.22, $p + q \in \mathcal{P}_o$ and $\theta(p)\theta(1 - q) = 0$ by Theorem 2.9. Hence $\tilde{\theta}(p)\tilde{\theta}(q) = 0$, and therefore $\tilde{\theta}$ preserves orthogonality of projections.

Let $q \in \widetilde{\mathcal{P}(A)}$ and $u \in \mathcal{U}(A)$. If $q \in \mathcal{P}_e$ (respectively $q \in \mathcal{P}_o$). Then by Proposition 2.7(i), $uqu^* \in \mathcal{P}_e$ (respectively $uqu^* \in \mathcal{P}_o$). From Proposition 2.1(i), it then follows that $\tilde{\theta}$ preserves unitary equivalence. \square

3. Oddly decomposable C^* -algebras

Let (A, B, φ, θ) be as in (2.4) and let \mathfrak{C} be the equivalence relation on $\widetilde{\mathcal{P}(A)}$ introduced in Section 2. We now introduce a sufficient condition on the unital, simple C^* -algebra A , such that $\widetilde{\mathcal{P}(A)}/\mathfrak{C}$ has at most two elements.

Definition 3.1. Let A be a C^* -algebra. A is said to be oddly decomposable if for every pair $p, q \in \widetilde{\mathcal{P}}(A)$ there is an odd integer $n \geq 3$ and a decomposition of q as a sum $q = \sum_{i=1}^n r_i$ of pairwise non-zero orthogonal projections r_i of A , such that each r_i is unitarily equivalent to some projection $r'_i < p$.

Remark 3.2. Let (A, B, φ, θ) be as in (2.4). If A is oddly decomposable, then A satisfies the condition of Proposition 2.13, hence $\widetilde{\mathcal{P}}(A)/\mathfrak{C}$ has at most two elements. More precisely, for any projection $p \in \widetilde{\mathcal{P}}(A)$, we have $c(\widetilde{\mathcal{P}}(A)) \subseteq \{c_p, \bar{c}_p\}$.

Proposition 3.3. Let (A, B, φ, θ) be as in (2.4), and suppose that A is oddly decomposable. Then for any projections $p, q \in \widetilde{\mathcal{P}}(A)$, there is a non-zero projection $r \in A$ for which $r < p$ and $r \mathfrak{C} q$.

Proof. Since A is oddly decomposable, there is an odd integer $n \geq 3$ and a decomposition of q as a sum $q = \sum_{i=1}^n r_i$ of pairwise non-trivial orthogonal projections such that r_i is unitarily equivalent to some projection $r'_i < p$. By Corollary 2.19, $q \mathfrak{C} r_j$ for some j ; hence $q \mathfrak{C} r'_j$. \square

Lemma 3.4. Let (A, B, φ, θ) be as in (2.4), and suppose that A is oddly decomposable. Let $p \in \widetilde{\mathcal{P}}(A)$. Suppose that p_1 and p_2 are two orthogonal projections such that $p_1 \mathfrak{C} p$, $p_2 \mathfrak{C} p$, $p_1 + p_2 \neq 1$ and $\theta(p_1)\theta(p_2) = 0$. Then θ preserves orthogonality on all projections which are \mathfrak{C} -equivalent to p .

Proof. let r and s be two orthogonal projections such that $r \mathfrak{C} p$ and $s \mathfrak{C} p$, and let us show that $\theta(r)\theta(s) = 0$. As $\theta(1 - r) = 1 - \theta(r)$, we can assume that $r + s \neq 1$, by Lemma 2.2. By Lemma 2.15, it is enough to show that there exist two orthogonal projections y, z which are \mathfrak{C} -equivalent to p , $y + z < 1 - (r + s)$ and $\theta(y)\theta(z) = 0$.

As A is oddly decomposable, there exist n , with n odd, pairwise orthogonal projections x'_1, \dots, x'_n , such that for each $1 \leq i \leq n$, x'_i is unitarily equivalent to some projection $x_i < 1 - (r + s)$, and

$$\sum_{i=1}^n x'_i = 1 - (p_1 + p_2).$$

By Corollary 2.19, we have $x'_i \mathfrak{C} 1 - (p_1 + p_2)$ for some i . To simplify notation set $x'_i = x'$ and $x_i = x$. By Lemma 2.2 and Proposition 2.7, then $x \mathfrak{C} (p_1 + p_2)$. As $\theta(p_1)\theta(p_2) = 0$ and $c_{p_1} = c_{p_2}$, then by Theorem 2.9, $c_{p_1} = c_{p_2} = c_{p_1+p_2}$. Hence, $x \mathfrak{C} p$, and by construction $x < 1 - (r + s)$.

Using the same argument as in the last paragraph, with x' replacing $1 - (r + s)$, there is a non-zero projection y' , with $y' < x'$ and $y' \mathfrak{C} p$. Again replacing x' by $x' - y'$, there is a non-zero projection z' , with $z' < x' - y'$, and $z' \mathfrak{C} p$.

As $\theta(p_1)\theta(p_2) = 0$, by Lemma 2.15, applied to $\{y', z', p_1, p_2\}$, we have that $\theta(y')\theta(z') = 0$.

Since x' and x are unitarily equivalent, let $u \in \mathcal{U}(A)$ be such that $x = ux'u^*$. Set $y = uy'u^*$ and $z = uz'u^*$. These z and y are \mathfrak{C} -equivalent to p by Proposition 2.7 and $\theta(y)\theta(z) = 0$.

As y and z are orthogonal projections, with

$$y + z < x < 1 - (r + s),$$

this finishes the proof of Lemma 3.4. \square

By contraposition, we also have

Corollary 3.5. *Let (A, B, φ, θ) be as in (2.4) and suppose that A is oddly decomposable. Let $p \in \widetilde{\mathcal{P}}(A)$. If p_1 and p_2 are two orthogonal projections \mathfrak{E} -equivalent to p such that $p_1 + p_2 \neq 1$ and $\theta(1 - p_1)\theta(1 - p_2) = 0$, then θ flips orthogonality on all pairs of projections \mathfrak{E} -equivalent to p .*

Let (A, B, φ, θ) be as is in (2.4), and suppose that A is oddly decomposable. Then by Remark 3.2, $c(\widetilde{\mathcal{P}}(A)) \subseteq \{c_p, \bar{c}_p\}$, for some $p \in \widetilde{\mathcal{P}}(A)$. If all projections are \mathfrak{E} -equivalent to p , then by Remark 2.12 the map θ is an orthoisomorphism. Otherwise, we define

$$\mathcal{P}_{c_p} = \{q \in \widetilde{\mathcal{P}}(A); c_q = c_p\}$$

and

$$\mathcal{P}_{\bar{c}_p} = \{q \in \widetilde{\mathcal{P}}(A); c_q = \bar{c}_p\}.$$

Clearly, \mathcal{P}_{c_p} and $\mathcal{P}_{\bar{c}_p}$ form a partition of $\widetilde{\mathcal{P}}(A)$. We then have:

Proposition 3.6. *Let (A, B, φ, θ) be as in (2.4), and suppose that A is oddly decomposable. Let $p \in \widetilde{\mathcal{P}}(A)$. Then θ preserves orthogonality in \mathcal{P}_{c_p} (respectively in $\mathcal{P}_{\bar{c}_p}$) and flips orthogonality in $\mathcal{P}_{\bar{c}_p}$ (respectively \mathcal{P}_{c_p}).*

Proof. Let $p_1, p_2 \in \mathcal{P}_{c_p}$ be projections such that $p_1 p_2 = 0$ and $p_1 + p_2 \neq 1$. By Theorem 2.9, either $\theta(p_1)\theta(p_2) = 0$ or $\theta(1 - p_1)\theta(1 - p_2) = 0$. By Lemma 3.4 and Corollary 3.5, we have that θ either preserves or flips orthogonality on \mathcal{P}_{c_p} , and likewise on $\mathcal{P}_{\bar{c}_p}$. We show that θ does not preserve orthogonality on $\mathcal{P}_{\bar{c}_p}$ if it does so on \mathcal{P}_{c_p} . Suppose that θ preserves orthogonality on both \mathcal{P}_{c_p} and $\mathcal{P}_{\bar{c}_p}$. By Corollary 2.19 and the fact that A is oddly decomposable, there exists a projection $q \in \mathcal{P}_{\bar{c}_p}$ such that $q < 1 - p$. Since the image of c contains at most two elements, we know that either $c_{1-(p+q)} = \bar{c}_p = c_q$ or $c_{1-(p+q)} = c_p = \bar{c}_q$. Interchanging p and q if necessary, we can assume that $c_{1-(p+q)} = \bar{c}_p$. By Proposition 3.3, let x be a projection \mathfrak{E} -equivalent to q and $x < p$. Then

$$c_{p-x} = c_{x+q+(1-(p+q))} = \bar{c}_p.$$

So x and $p - x$ are orthogonal projections in $\mathcal{P}_{\bar{c}_p}$ such that

$$c_x = c_{p-x} = \bar{c}_{x+(p-x)} = \bar{c}_p.$$

As θ preserves orthogonality on \mathcal{P}_{c_p} , $c_x = c_{p-x} = c_{x+(p-x)}$. This implies that $c_p = \bar{c}_p$, so θ does not preserve orthogonality on $\mathcal{P}_{\bar{c}_p}$ if it does on \mathcal{P}_{c_p} .

Now let us show that θ does not flip orthogonality on \mathcal{P}_{c_p} if it does on $\mathcal{P}_{\bar{c}_p}$. Suppose that θ flips orthogonality on both \mathcal{P}_{c_p} and $\mathcal{P}_{\bar{c}_p}$. By Proposition 3.3 there is a projection $q \in \mathcal{P}_{\bar{c}_p}$ such that $q < 1 - p$. Without loss of generality suppose that $c_{1-(p+q)} = \bar{c}_p$. Again use Corollary 2.19 and the fact that A is oddly decomposable to find $x < p$ such that $c_x = c_p$. If $c_{p-x} \neq c_x$, then by Remark 3.2 we have $c_{p-x} = \bar{c}_x = c_q$, and so

$$c_x = c_{(p-x)+q+(1-(p+q))} = c_q = \bar{c}_x.$$

So x and $p - x$ are orthogonal projections in \mathcal{P}_{c_p} such that $c_x = c_{p-x} = \bar{c}_p$, hence θ does not flip orthogonality on both \mathcal{P}_{c_p} and $\mathcal{P}_{\bar{c}_p}$. \square

Therefore, combining Proposition 3.6 and Theorem 2.21, we have the following main result:

Theorem 3.7. *Let (A, B, φ, θ) be as in (2.4). If A is oddly decomposable, then φ induces an orthoisomorphism between the sets of projections $\mathcal{P}(A)$ and $\mathcal{P}(B)$, which preserves the unitary equivalence of projections.*

4. The case of simple AH-algebras

4.1. From orthoisomorphism to a K_0 -order isomorphism

In this subsection, we prove that an (abstract) isomorphism between the unitary groups of a class of finite C^* -algebras of real rank zero (including the simple AH-algebras of slow dimension growth) induces an isomorphism between their ordered K_0 -groups. In particular, we have that if A and B are either two simple unital AF-algebras, or two irrational rotation algebras, then A is $*$ -isomorphic to B if and only if their unitary groups are isomorphic (as abstract groups).

Let \mathcal{F} denote the class of simple, unital, separable C^* -algebras of real rank zero with cancellation and

$$\mathcal{F}_1 = \{A \in \mathcal{F}; K_0(A) \text{ is noncyclic and weakly unperforated}\}.$$

Recall that if $A \in \mathcal{F}$, then A has stable rank one by [3], Corollary 6.5.7 and therefore is stably finite. Hence (see [17], Theorem 3.3.18) $(K_0(A), K_0(A)_+)$ is a (simple) ordered group with Riesz interpolation property.

If (G, G_+) is an ordered group, then (see [10], Chapter 7) recall that a scale is a subset Γ of G_+ , which is generating, hereditary and directed, i.e.,

S1. For each $a \in G_+$, there exist $a_1, \dots, a_r \in \Gamma$ with $a = a_1 + a_2 + \dots + a_r$.

S2. If $0 \leq a \leq b \in \Gamma$, then $a \in \Gamma$.

S3. Given $a, b \in \Gamma$, there exists $c \in \Gamma$ with $a, b \leq c$.

Following [10], a scaled dimension group G is a dimension group with a distinguished scale denoted $\Gamma = \Gamma(G)$, and a homomorphism of scaled dimension groups $f : G \rightarrow G'$ is a *contraction* if $f(\Gamma) \subseteq \Gamma'$.

The scale Γ of a scaled dimension group G has a partially defined addition; in fact $a \geq b$ in Γ if and only if $a = b + c$ for some $c \in \Gamma$. If Γ and Γ' are scales of two scaled dimension groups G and G' , then (see [10], p. 45) a map $f : \Gamma \rightarrow \Gamma'$ is a scale homomorphism (respectively scale isomorphism) if $a = b + c$ in Γ implies that (respectively is equivalent to) $f(a) = f(b) + f(c)$ in Γ' .

Then we have (see [10], Lemma 7.3 and Corollary 7.4)

Proposition 4.1. (See [10].) *Let G and G' be two ordered groups with Riesz interpolation. Any scale homomorphism $f : \Gamma(G) \rightarrow \Gamma(G')$ extends to a unique contraction $\tilde{f} : G \rightarrow G'$. If f is a scale isomorphism, then \tilde{f} is an isomorphism of the scaled ordered groups G and G' .*

Recall that if A is a unital C^* -algebra with cancellation, then

$$\Sigma(A) = \{[p]; p \text{ is a projection in } A\} \subseteq K_0(A)$$

is the closed interval $[0, [1_A]] = \{x \in K_0(A)_+; x \leq [1_A]\}$. Moreover, if $A \in \mathcal{F}$, then $\Sigma(A)$ is a scale of the ordered group $(K_0(A), K_0(A)_+)$ with Riesz interpolation.

Proposition 4.2. *If $A \in \mathcal{F}_1$, then A is oddly decomposable.*

Proof. Let p, q be two non-trivial projections in A . As $[q] > 0$ in the noncyclic simple ordered group $K_0(A)$, there exists by [15], Lemma 14.5, $y \in K_0(A)_+$ with

$$0 < y < [q],$$

and as y is an order unit, $[p] \leq (2k+1)y$, for some $k \in \mathbb{Z}^+$. By the Riesz interpolation property, there are $a_1, \dots, a_{2k+1} \in K_0(A)_+$ such that

$$[p] = a_1 + \dots + a_{2k+1}, \quad \text{with } 0 < a_i \leq y < [q].$$

By [17], Lemma 3.4.2, there exist mutually orthogonal projections $p_1, p_2, \dots, p_{2k+1} \in A$ such that

$$p = p_1 + p_2 + \dots + p_{2k+1} \quad \text{and} \quad [p_i] = a_i < [q].$$

As $K_0(A)$ is weakly unperforated, by [3], Corollary 6.9.2, we have $p_i \preceq q$, for $1 \leq i \leq 2k+1$. Hence, as A has cancellation, each p_i is unitarily equivalent to some $q_i \leq q$. \square

We can now prove the main result of this section.

Theorem 4.3. *Let A and B be two C^* -algebras in \mathcal{F}_1 . If $\mathcal{U}(A)$ and $\mathcal{U}(B)$ are isomorphic, then $K_0(A)$ and $K_0(B)$ are isomorphic as scaled ordered groups.*

Proof. Let $\tilde{\theta} : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ be the orthoisomorphism preserving unitary equivalence of projections given by Proposition 4.2 and Theorem 3.7. If $[p] = [q]$ in $K_0(A)$, then as A has cancellation, p and q (respectively $1-p$ and $1-q$) are Murray von Neumann equivalent, and then p is unitarily equivalent to q ; hence $[\tilde{\theta}(p)] = [\tilde{\theta}(q)]$ in $K_0(B)$. Therefore, we get a map $\tilde{\theta}_* : \Sigma(A) \rightarrow \Sigma(B)$ given by $\tilde{\theta}_*([p]) = [\tilde{\theta}(p)]$, for $p \in \mathcal{P}(A)$. Let us check that $\tilde{\theta}_*$ is a scale homomorphism. Let x, y and $z \in \Sigma(A)$ with $x + y = z$. Let $p, q \in \mathcal{P}(A)$ be such that $x = [p]$ and $y = [q]$. If $z = [1_A]$, then $[p] = [1_A] - [q] = [1_A - q]$. As $\tilde{\theta}(1_A - p) = 1_B - \tilde{\theta}(p)$, we have:

$$\begin{aligned} \tilde{\theta}_*([1_A]) &= \tilde{\theta}_*([q + 1 - q]) = [\tilde{\theta}(q + 1 - q)] \\ &= [\tilde{\theta}(q) + \tilde{\theta}(1 - q)] = [\tilde{\theta}(q) + (1_B - \tilde{\theta}(q))] \\ &= [\tilde{\theta}(q)] + [1_B - \tilde{\theta}(q)] = [\tilde{\theta}(q)] + [\tilde{\theta}(1_A - q)] \\ &= \tilde{\theta}_*(y) + \tilde{\theta}_*([1_A - q]) = \tilde{\theta}_*(y) + \tilde{\theta}_*(x). \end{aligned}$$

If $[p] + [q] = z < [1_A]$, then $[p] < [1_A - q]$ as A has cancellation and $K_0(A)$ is weakly unperforated, therefore by [3], Corollary 6.9.2, p is Murray-von Neumann equivalent to a subprojection q_1 of q . Hence

$$\begin{aligned}\tilde{\theta}_*(z) &= \tilde{\theta}_*([p] + [q]) = \tilde{\theta}_*([q_1] + [q]) = \tilde{\theta}_*([q_1 + q]) \\ &= [\tilde{\theta}(q_1 + q)] = [\tilde{\theta}(q_1) + \tilde{\theta}(q)] = [\tilde{\theta}(q_1)] + [\tilde{\theta}(q)] \\ &= \tilde{\theta}_*([q_1]) + \tilde{\theta}_*([q]) = \tilde{\theta}_*(x) + \tilde{\theta}_*(y).\end{aligned}$$

As $\tilde{\theta} : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ is an orthoisomorphism, its inverse induces a scale homomorphism $(\tilde{\theta}^{-1})_*$ from $\Sigma(B)$ to $\Sigma(A)$ such that $(\tilde{\theta}_*)^{-1} = (\tilde{\theta}^{-1})_*$. Hence by Proposition 4.1, $\tilde{\theta}_*$ is a scale isomorphism. \square

Recall that a C^* -algebra A is an AH-algebra if it can be realized as an inductive limit $\varinjlim A_n$, where $(A_n)_{n \geq 1}$ is a sequence of C^* -algebras of the form

$$A_n = \bigoplus_{i=1}^{r_n} p_{n,i} \mathbb{M}_{k_{n,i}}(C(X_{n,i})) p_{n,i} \quad (15)$$

where $r_n, k_{n,i}$ are natural numbers, $X_{n,i}$ are finite, connected CW-complexes and $p_{n,i}$ is a (non-zero) projection in $\mathbb{M}_{k_{n,i}}(C(X_{n,i}))$.

An AH-algebra A is said to have slow dimension growth (SDG), if A can be realized as the inductive limit of a sequence of C^* -algebras $(A_n)_{n \geq 1}$ as in Eq. (15) with

$$\lim_{n \rightarrow \infty} \max \{ \dim(X_{n,i})/d_{n,i} \mid i = 1, 2, \dots, r_n \} = 0,$$

where $d_{n,i} = \text{rank}(p_{n,i})$.

By [4], Theorem 1, every simple AH-algebra A of slow dimension growth has stable rank one, and if moreover A is unital and has real rank zero, then by [12], Theorem 4.18, $(K_0(A), K_0(A)^+)$ is a simple, weakly unperforated, ordered group with the Riesz interpolation property.

Therefore, every infinite dimensional, simple, unital AH-algebra of slow dimension growth belongs to the class \mathcal{F}_1 . Then Theorem 4.3 implies:

Corollary 4.4. *If A and B are simple, unital AH-algebras of slow dimension growth and of real rank zero, with isomorphic unitary groups (as abstract groups), then $(K_0(A), K_0(A)^+, [1_A])$ and $(K_0(B), K_0(B)^+, [1_B])$ are order isomorphic by a map preserving the distinguished order units.*

Recall that (see [19], Definition 2.4.5, for example) that a separable C^* -algebra belongs to the UCT-class \mathcal{N} if it is KK -equivalent to an abelian C^* -algebra. Using H. Lin's characterization of C^* -algebras of tracial topological rank zero (TAF-algebra), in [17] (or for example, see Theorem 3.3.5 in [19]), we can also state Corollary 4.4 as follows.

Corollary 4.5. *Let A and B be two simple, unital, nuclear, separable TAF-algebras of real rank zero, belonging to the UCT-class \mathcal{N} . If $\mathcal{U}(A)$ and $\mathcal{U}(B)$ are isomorphic (as abstract groups), then*

$$(K_0(A), K_0(A)_+, [1_A]) \quad \text{and} \quad (K_0(B), K_0(B)_+, [1_B])$$

are order isomorphic by a map preserving the distinguished order units.

In case VII of Lemma 13 of [9], H. Dye shows that if M and N are two finite dimensional von Neumann factors whose unitary groups are isomorphic (as abstract groups), then M and N are isomorphic (as von Neumann algebras). With this result and Corollary 4.4, we have:

Corollary 4.6. *If A and B are simple, unital AF-algebras with isomorphic unitary groups (as abstract groups), then A and B are isomorphic as C^* -algebras.*

Recall that by [11] every irrational rotation algebra A_θ is an $A\mathbb{T}$ -algebra of real rank zero. If A_θ and A_η are two irrational rotation algebras, with isomorphic unitary groups, then by Corollary 4.4, $\theta = \pm\eta \bmod \mathbb{Z}$ and therefore (see for example [8], Corollary VI, 5.3), we have:

Corollary 4.7. *Two irrational rotation algebras A_θ and A_η are isomorphic if and only if their unitary groups are isomorphic (as abstract groups).*

Let (X, φ) be a Cantor minimal system, i.e. X is the Cantor set and φ is a minimal self-homeomorphism of X . In [18], I. Putnam showed that the crossed-product C^* -algebra $C^*(X, \varphi) = C(X) \rtimes_\varphi \mathbb{Z}$ is then a simple $A\mathbb{T}$ -algebra of real rank zero. Let $K^0(X, \varphi)$ denote the ordered abelian group

$$C(X, \mathbb{Z}) / \{f - f \circ \varphi^{-1}; f \in C(X, \mathbb{Z})\},$$

recall that $K_0(C^*(X, \varphi))$ is order isomorphic to $K^0(X, \varphi)$ with order unit, and $K_1(C^*(X, \varphi)) \cong \mathbb{Z}$. By [13], Theorems 2.1 and 4.3, we then get:

Corollary 4.8. *Let X be the Cantor set and let φ and ψ be minimal homeomorphisms on X . Then the following conditions are equivalent:*

- (i) *The unitary groups $\mathcal{U}(C^*(X, \varphi))$ and $\mathcal{U}(C^*(X, \psi))$ are isomorphic (as abstract groups),*
- (ii) *$C^*(X, \varphi) \cong C^*(X, \psi)$,*
- (iii) *$K^0(X, \varphi) \cong K^0(X, \psi)$ as ordered abelian groups with order units,*
- (iv) *φ and ψ are strong orbit equivalent.*

4.2. From a topological unitary group isomorphism to a C^* -isomorphism

For simple AH-algebras of real rank zero, let us recall the classification theorem, proved independently by Gong in [14] and Dadarlat in [7], and whose proof uses Elliott–Gong’s classification in [12] (see for example [19], Theorem 3.3.1).

Theorem 4.9. *Let A and B be simple, unital AH-algebras of slow dimension growth and of real rank zero. Then A is isomorphic to B if and only if*

$$(K_0(A), K_0(A)^+, [1_A]) \simeq (K_0(B), K_0(B)^+, [1_B]), \quad K_1(A) \simeq K_1(B).$$

Recall (see for example [2], Corollary 7.1.4) that if A is a simple, unital C^* -algebra with stable rank one, then the natural homomorphism

$$\mu : \mathcal{U}(A)/\mathcal{U}(A)_0 \rightarrow K_1(A)$$

is an isomorphism. Therefore if $\varphi : \mathcal{U}(A) \rightarrow \mathcal{U}(B)$ is a topological isomorphism between the unitary groups of two simple, unital C^* -algebras with stable rank one, then φ induces an isomorphism between $K_1(A)$ and $K_1(B)$.

By Theorem 4.9, and Corollary 4.4, we have:

Theorem 4.10. *Let A and B be two simple, unital AH-algebras of slow dimension growth and of real rank zero. Then A and B are isomorphic if and only if their unitary groups are topologically isomorphic.*

5. The case of Kirchberg algebras

5.1. From orthoisomorphism to K_0 -isomorphism

In this subsection, we show that an isomorphism between the unitary groups of simple, unital, purely infinite C^* -algebras induces an isomorphism between their K_0 -groups.

To prove Theorem 5.2, we will need the following result (see the remark after [6], Proposition 1.5):

Proposition 5.1. *Given projections p, q in a simple C^* -algebra A , with q infinite, there is a projection p' in A such that $p \sim p'$ and $q - p'$ is infinite.*

Recall also (see [6], p. 187, for example) that if A is a purely infinite simple C^* -algebra, then each non-zero projection in A is infinite and that

$$K_0(A) = \{[p]; p \in \mathcal{P}(A), p \neq 0\}.$$

Moreover, if A is unital, then as 1 is an infinite projection and therefore equivalent to a projection $q < 1$, we have:

$$K_0(A) = \{[p]; p \in \widetilde{\mathcal{P}(A)}\}. \quad (16)$$

Theorem 5.2. *Every simple, unital purely infinite C^* -algebra A is oddly decomposable.*

Proof. Let p, q be two non-trivial projections of A . As both are infinite projections in A , there exist a projection $q' \in A$, with $q \sim q' < q$ and by Proposition 5.1, a projection p_1 such that $q' \sim p_1 < p$. Again by 5.1, there exist a projection p_2 with $q' \sim p_2 < p - p_1$, and a projection r such that $p - p_1 - p_2 \sim r \leq q'$. Hence, $p = p_1 + p_2 + (p - p_1 - p_2)$ is the sum of three orthogonal projections, each of them is equivalent to a proper subprojection of q , and therefore by [3], Corollary 6.11.9, unitarily equivalent to a subprojection of q . This shows that A is oddly decomposable. \square

Theorem 5.3. *If A and B are two unital, simple, purely infinite C^* -algebras, whose unitary groups are isomorphic (as abstract groups), then there is an isomorphism from $K_0(A)$ to $K_0(B)$, sending $[1_A]$ to $[1_B]$.*

Proof. Let $\theta : \mathcal{P}(A) \rightarrow \mathcal{P}(B)$ be an orthoisomorphism preserving the unitary equivalence of projections, whose existence follows from Theorems 3.7 and 5.2.

Recall that if p and q are two non-trivial projections of A , then by [3], Corollary 6.11.9, $[p] = [q]$ in $K_0(A)$ if and only if p and q are unitarily equivalent. Therefore, we can define a map $\Theta : K_0(A) \rightarrow K_0(B)$ by $\Theta([p]) = [\theta(p)]$, for all $p \in \widehat{\mathcal{P}}(A)$. If $p, q \in \widehat{\mathcal{P}}(A)$ and $[\theta(p)] = [\theta(q)]$ in $K_0(B)$, then $\theta(p) \sim_u \theta(q)$, which implies $p \sim_u q$ and therefore $[p] = [q]$ in $K_0(A)$. Hence Θ is a one-to-one and is onto by definition.

Let r be a fixed non-trivial projection of A . If $p, q \in \widehat{\mathcal{P}}(A)$, then by 5.1, there are projections p' and q' in A such that $p \sim p' \leq r$ and $q \sim q' \leq 1 - r$. Therefore, $p'q' = 0$ and $p \sim_u p'$ and $q \sim_u q'$. Similarly $\theta(p')\theta(q') = 0$ and $\theta(p) \sim_u \theta(p')$ and $\theta(q) \sim_u \theta(q')$. Hence, we have:

$$\begin{aligned} \Theta([p] + [q]) &= \Theta([p'] + [q']) \\ &= \Theta([p' + q']) \\ &= [\theta(p') + \theta(q')] \\ &= [\theta(p')] + [\theta(q')] \\ &= [\theta(p)] + [\theta(q)] \\ &= \Theta([p]) + \Theta([q]), \end{aligned}$$

which shows that Θ is an isomorphism.

As $[I_A] = [r + 1_A - r] = [r] + [1_A - r]$ and as $\theta(1_A - r) = 1_B - \theta(r)$, then

$$\begin{aligned} \Theta([1_A]) &= \Theta([r] + [1_A - r]) \\ &= [\theta(r)] + [\theta(1_A - r)] \\ &= [\theta(r)] + [1_B - \theta(r)] \\ &= [\theta(r) + (1_B - \theta(r))] \\ &= [1_B]. \quad \square \end{aligned}$$

In [6], J. Cuntz proved that for $2 \leq n < \infty$, $K_0(\mathcal{O}_n) \cong \mathbb{Z}/(n-1)\mathbb{Z}$ and $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$. Hence, we have:

Corollary 5.4. *Two Cuntz algebras are isomorphic if and only if their unitary groups are isomorphic (as abstract groups).*

5.2. From a unitary to a K_1 -group isomorphism

Let C be a unital, purely infinite, simple C^* -algebra. If $\mathcal{U}_0(C)$ denotes the connected component of the unitary group $\mathcal{U}(C)$, then by [6], Theorem 1.9, $K_1(C) \cong \mathcal{U}(C)/\mathcal{U}_0(C)$. Moreover, by [16], Theorem 3.8, $\mathcal{U}_0(C)$ is generated by the self-adjoint unitaries of C .

Theorem 5.5. *Let A and B be two simple, unital, purely infinite C^* -algebras. If φ is an isomorphism between $\mathcal{U}(A)$ and $\mathcal{U}(B)$, then the groups $K_1(A)$ and $K_1(B)$ are isomorphic.*

Proof. Let $\varphi : \mathcal{U}(A) \rightarrow \mathcal{U}(B)$ be an isomorphism. As φ maps self-adjoint unitaries onto self-adjoint unitaries, by [16], Theorem 3.8, $\varphi(\mathcal{U}_0(A)) = \mathcal{U}_0(B)$. Hence φ induces an isomorphism from $K_1(A)$ onto $K_1(B)$. \square

Recall (see [19], Definition 4.3.1) that a Kirchberg algebra is a purely infinite, simple, nuclear, separable C^* -algebra, and that the following result of Kirchberg and Phillips classifies them:

Theorem 5.6. *(See [19], Theorem 8.4.1.) Let A and B be two unital Kirchberg algebras belonging to the UCT-class \mathcal{N} . Then A and B are $*$ -isomorphic if and only if there are isomorphisms $\alpha_0 : K_0(A) \rightarrow K_0(B)$ and $\alpha_1 : K_1(A) \rightarrow K_1(B)$ with $\alpha_0([1_A]) = [1_B]$.*

Thanks to Theorems 5.3 and 5.5, we then get:

Corollary 5.7. *Let A and B be two unital Kirchberg algebras belonging to the UCT-class \mathcal{N} . Then A and B are isomorphic if and only if their unitary groups are isomorphic (as abstract groups).*

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